Modelling non-stationarity in asymptotically independent extremes (in review)

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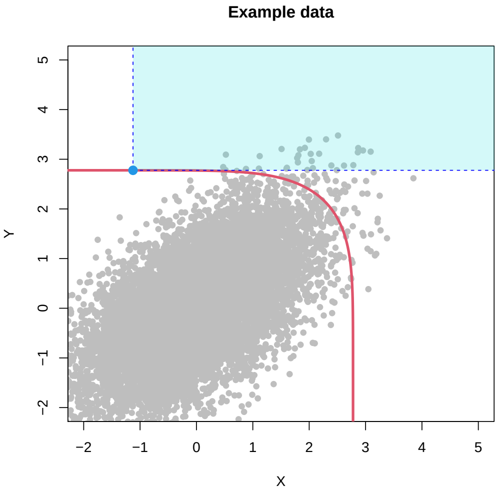
Motivation - return curves

- My PhD: estimating **return curves** at extreme values.
- Given two variables,

$$\mathrm{RC}(p) := \{(x, y) \in \mathbb{R}^2 \mid \Pr(X > x, Y > y) = p\},$$

  where p is very small.
- Provides a summary of **extremal dependence**.
- Return level **extension**.

Motivation - return curves



Motivation - return curves

- In practice, we wish to estimate return curves for **combinations of environmental variables** (temperature, relative humidity, wind speed etc.).
- Such variables exhibit **non-stationarity**.

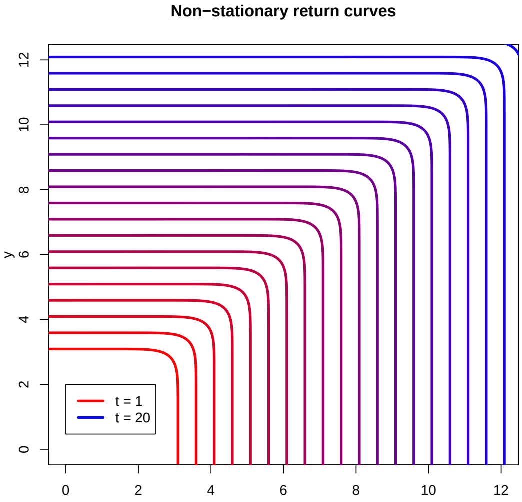
Motivation - return curves

- Return curves **lack meaning** in the non-stationary setting, motivating an extended definition.
- Given $\{X_t, Y_t\}$ with **covariates** $\mathbf{Z}_t, \ t \in \{1, 2, \ldots, T\}$,

$$\mathrm{RC}_{\mathbf{z}_t}(p) := \{(x, y) \in \mathbb{R}^2 \mid \Pr(X_t > x, Y_t > y \mid \mathbf{Z}_t = \mathbf{z}_t) = p\}.$$

- **End goal**: estimating non-stationary return curves.

Motivation - return curves



Multivariate Extremes

- To estimate return curves, we require **models** for evaluating the **joint tail** of (X, Y).
- **Frameworks** for assessing **extremal dependence**.
- **Classification** of extremal dependence given by the coefficient χ:

$$\chi = \lim_{u \to 1} \Pr(F_Y(Y) > u \mid F_X(X) > u) \in [0, 1].$$

- $\chi = 0 \Rightarrow$ **asymptotic independence**.
- $\chi > 0 \Rightarrow$ **asymptotic dependence**.

Multivariate Extremes

- Classical models based on framework of **multivariate regular variation**.
- Given a random vector (X, Y) with standard Fréchet margins, define $R := X + Y$ and $W := X/(X + Y)$:

$$\lim_{r \to \infty} \Pr(W \in B, R > sr \mid R > r) = H(B)s^{-1}, \ s > 1.$$

- H is termed the **spectral measure**

Multivariate Extremes

- Downside: multivariate regular variation is only **suitable** in the case of **asymptotic dependence**.
- Extremal dependence structure **unknown** in practice.
- Motivates models that can capture **both** extremal dependence regimes.

Multivariate Extremes

- **First model** proposed in Ledford and Tawn (1996).
- Given random vector (X, Y) on standard exponential margins,

$$\Pr(X > u, Y > u) = \Pr(\min(X, Y) > u) \to L(e^u)\exp(-u/\eta),$$

  as $u \to \infty$, with L slowly varying and $\eta \in (0, 1]$.
- $\eta = 1 \Rightarrow$ **asymptotic dependence**.
- $\eta < 1 \Rightarrow$ **asymptotic independence**.
- Equal marginal growth rates $\Rightarrow$ **limited applicability**.

Multivariate Extremes

- Ledford and Tawn (1996) model was **extended** in Wadsworth and Tawn (2013).
- Given any **ray** $w \in [0, 1]$,

$$\Pr(X > wu, Y > (1-w)u) =$$
$$\Pr\left(\min\left\{\frac{X}{w}, \frac{Y}{1-w}\right\} > u\right) \to L(e^u \mid w)\exp(-\lambda(w)u),$$

as $u \to \infty$, with L slowly varying.

Multivariate Extremes

- $\lambda(w) \geq \max(w, 1-w)$ is termed the **angular dependence function** (ADF).
- **Summarises** the joint tail behaviour.
- Captures **both** asymptotic dependence (lower bound) and asymptotic independence.
- Ledford and Tawn (1996) recovered when $w = 0.5 \Rightarrow \eta = 1/(2\lambda(0.5))$.
- Allows evaluation of extremal dependence in **all regions**.

Multivariate Extremes

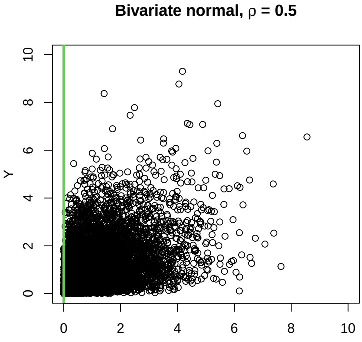
Define $K_w := \min\left\{\frac{X}{w}, \frac{Y}{1-w}\right\}$.

$$\Pr\left(K_w > u + v | K_w > u\right) \to \exp(-\lambda(w)v),$$

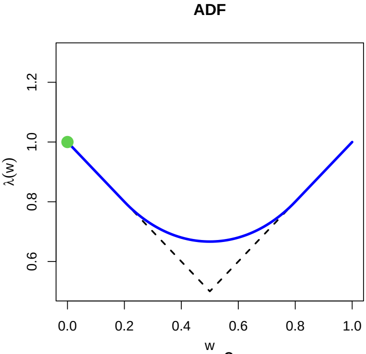
as $u \to \infty$ for any $v > 0$.

Simplified: $(K_w - u \mid K_w > u) \sim \text{Exp}(\lambda(w))$ for all $w \in [0, 1]$.

Multivariate Extremes

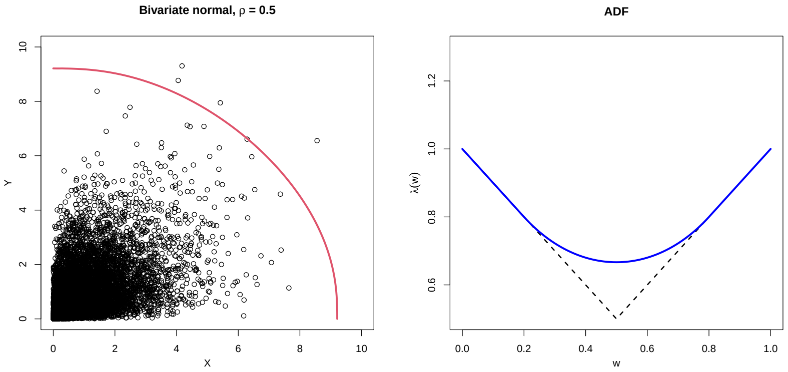


w = 0

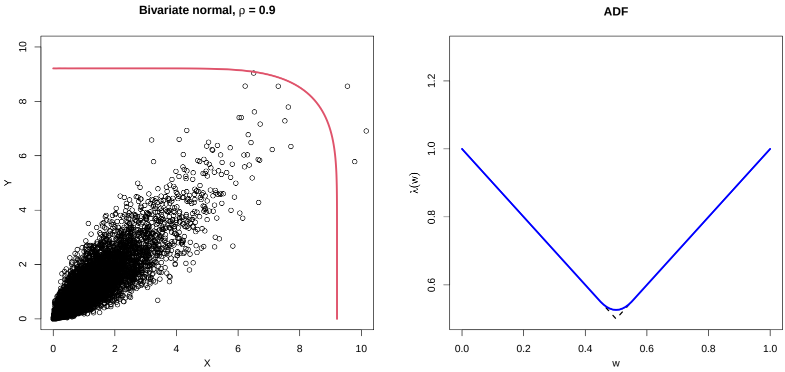


w = 0

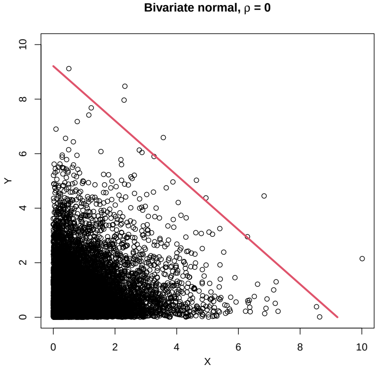
Multivariate Extremes

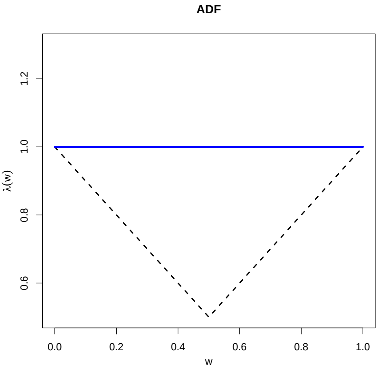


Multivariate Extremes

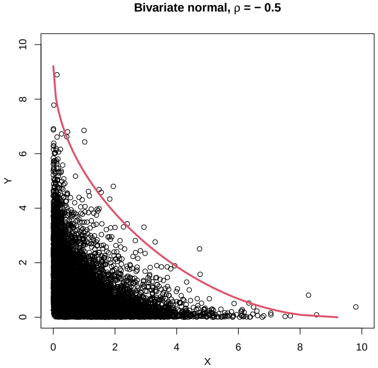


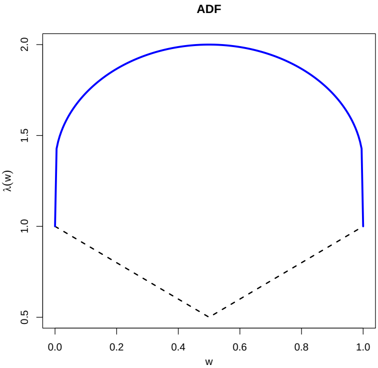
Multivariate Extremes





Multivariate Extremes

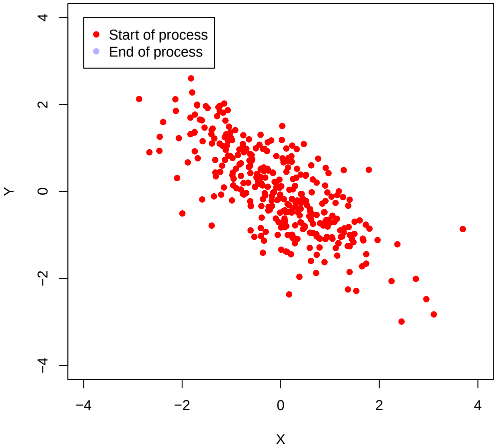




Non-stationarity

- Let $\{X_t, Y_t\}$ with $\mathbf{Z}_t,\ t \in \{1, 2, \ldots, T\}$, denote a **non-stationary process**.
- Two forms of non-stationarity can exist.
  1. Trends in marginal extremes - tail of X_t (Y_t). **Well studied.**
  2. Trends in extremal dependence. **Sparse literature.**
- Must account for **both forms** in our estimation procedure.
- We focus on **second problem** and present a **novel** modelling technique.

Non-stationarity



Non-stationarity

- **Few approaches** for capturing non-stationarity in extremal dependence.
- **Almost all** developed using multivariate regular variation framework.
- Asymptotically independent case **not well studied**.

Non-stationarity

We propose an **non-stationary extension** to the Wadsworth and Tawn (2013) model.

With $\{X_t, Y_t\}$ on standard exponential margins and $w \in [0, 1]$:

1. Define $K_{w,t} := \min\left\{\frac{X_t}{w}, \frac{Y_t}{1-w}\right\}$.
2. Assume

$$\Pr\left(K_{w,t} > v + u \middle| K_{w,t} > u, \mathbf{Z}_t = \mathbf{z}_t\right) \to \exp(-\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t)v),$$

   as $u \to \infty$ for any $v > 0$ and $t \leq T$.

$\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t)$ is termed the **non-stationary** ADF.

Non-stationarity

Can estimate via quantile regression on $K_{w,t}$:

- Select **probabilities** $q_1 < q_2 < 1$ close to 1.
- For a **fixed** w, find u_1 and u_2 such that

$$\Pr\left(K_{w,t} \leq u_1 \middle| \mathbf{Z}_t = \mathbf{z}_t\right) = q_1$$
$$\Pr\left(K_{w,t} \leq u_2 \middle| \mathbf{Z}_t = \mathbf{z}_t\right) = q_2.$$

- As $q_1 \to 1$, $u_1 \to \infty$, so we have

$$\frac{1 - q_2}{1 - q_1} = \exp\{-\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t)(u_2 - u_1)\}$$

Non-stationarity

$$\frac{1-q_2}{1-q_1} = \exp\{-\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t)(u_2 - u_1)\}$$

Estimator given by

$$\hat{\lambda}(w \mid \mathbf{Z}_t = \mathbf{z}_t) = -\frac{1}{u_2 - u_1} \log\left(\frac{1-q_2}{1-q_1}\right)$$

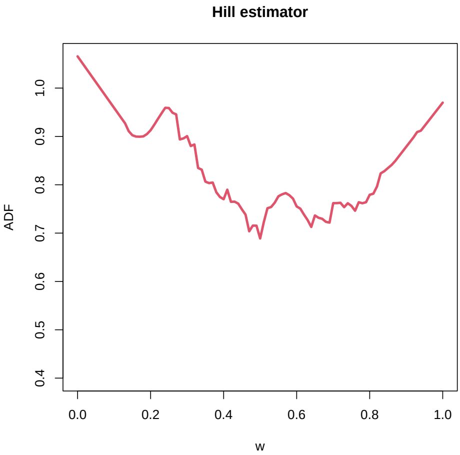
Non-stationarity

In practice, there are a few additional steps.

- **Averaging** over different values for q_1 and q_2. **Bias-variance trade-off.**
- **Smoothing**.
- Imposing **theoretical properties** of ADF.

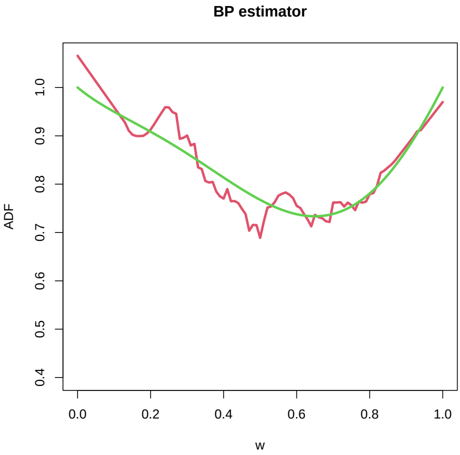
Non-stationarity

Estimator is **pointwise** - unrealistic.



Non-stationarity

Smooth using **Bernstein-Bézier polynomials** (Marcon et al., 2017).



See Murphy-Barltrop and Wadsworth (2022) for further details.

Non-stationarity

- We can use non-stationary ADFs to estimate return curves.

$$\Pr(X_t > x, Y_t > y \mid \mathbf{Z}_t = \mathbf{z}_t) = p$$

- Given any ray $w \in [0, 1]$ and t, define r as

$$r := -\frac{1}{\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t)} \log\left(\frac{p}{1-q}\right).$$

  with $q < 1 - p < 1$.
- Implies

$$\frac{p}{1-q} = \exp(-\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t) r).$$

Non-stationarity

- Let $(x, y) := (w(r+u), (1-w)(r+u))$, with u equal to the q-th quantile of $K_{w,t}$.
- We have

$$\begin{aligned} & \Pr(X_t > x, Y_t > y \mid \mathbf{Z}_t = \mathbf{z}_t) \\ = & \Pr(X_t/w > r+u, Y_t/(1-w) > r+u \mid \mathbf{Z}_t = \mathbf{z}_t) \\ = & \Pr(\min\{X_t/w, Y_t/(1-w)\} > r+u \mid \mathbf{Z}_t = \mathbf{z}_t) \\ = & \Pr(K_{w,t} > r+u \mid \mathbf{Z}_t = \mathbf{z}_t) \end{aligned}$$

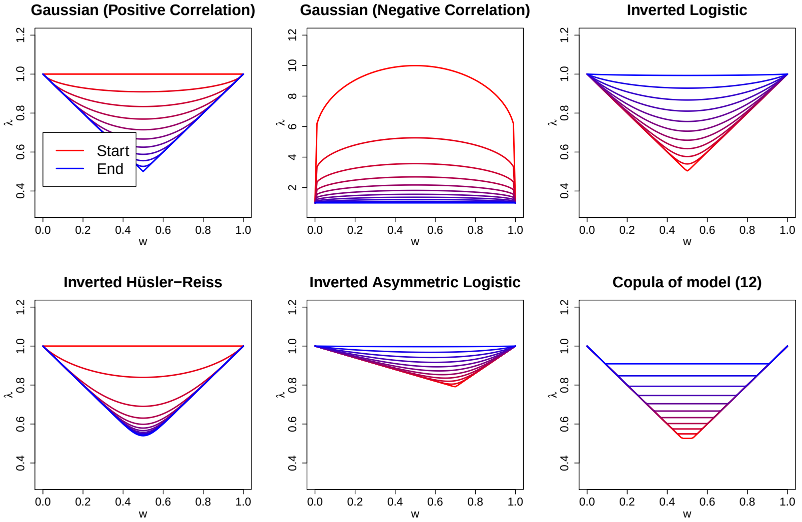
Non-stationarity

$$
\begin{aligned}
&\Pr(K_{w,t} > r + u \mid \mathbf{Z}_t = \mathbf{z}_t) \\
&= \Pr(K_{w,t} > r + u \mid K_{w,t} > u, \mathbf{Z}_t = \mathbf{z}_t) \\
&\quad \times \Pr(K_{w,t} > u \mid \mathbf{Z}_t = \mathbf{z}_t) \\
&\approx \exp\{-\lambda(w \mid \mathbf{Z}_t = \mathbf{z}_t) r\} \Pr(K_{w,t} > u \mid \mathbf{Z}_t = \mathbf{z}_t) \\
&= \frac{p}{1-q} \times 1 - q = p,
\end{aligned}
$$

So (x, y) is a point on the return curve $\mathrm{RC}_{\mathbf{z}_t}(p)$. Repeat for all w and t.

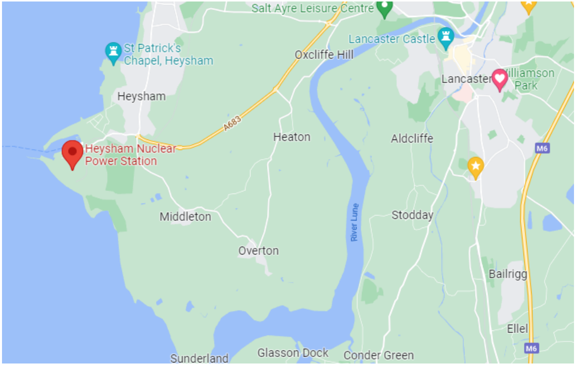
Simulation study

I won't bore you with the details.



Motivating example - UKCP18 data

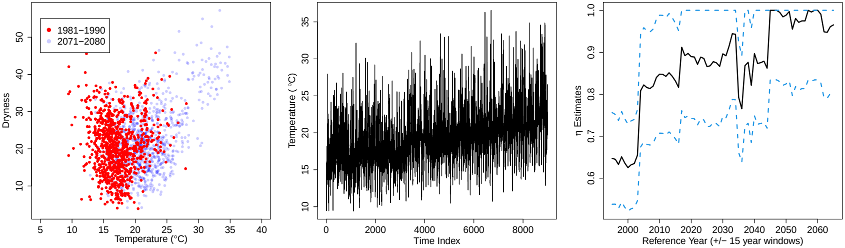
- Take the 1980-2080 **temperature** and **relative humidity** UKCP18 data for the nuclear site at Heysham, UK.
- We focus on **summer data only**.



Motivating example - UKCP18 data

- Suppose $0 < RH < 100$ represents relative humidity. We define a **'dryness'** variable: $Dr := 100 - RH$.
- **Combination** of high temperature and high dryness typical of drought-like conditions.
- This is a **concern** for nuclear regulators (Knochenhauer and Louko, 2004).
- **Understanding relationship** between the extremes could allow for better risk management.

Motivating example - UKCP18 data

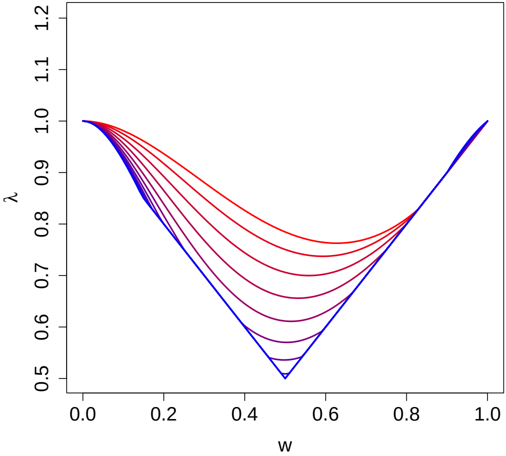


Motivating example - UKCP18 data

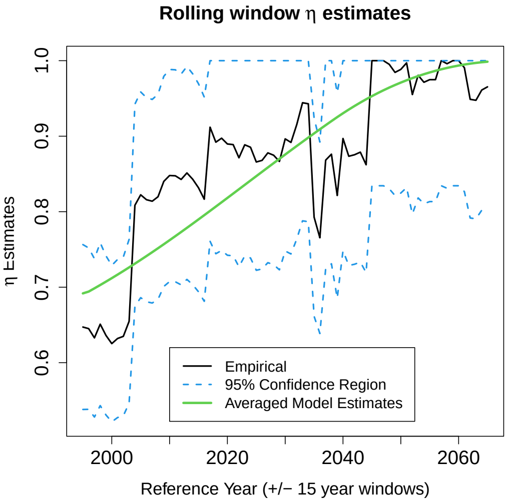
1. **Remove marginal trends** using methods proposed in Davison and Smith (1990) and Eastoe and Tawn (2009).
2. **Transform** data to **exponential margins**.
3. **Estimate** non-stationary ADF.
4. **Calculate** return curve estimates up to the **year 2080**.

Motivating example - UKCP18 data

Red = start, blue = end.



Motivating example - UKCP18 data

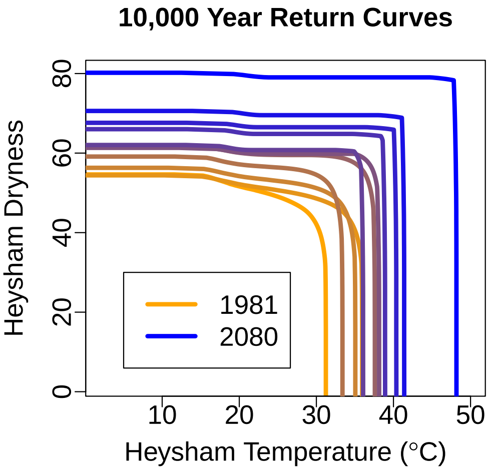


(Model estimate given by $\hat{\eta}_t = [2\hat{\lambda}(0.5 \mid \mathbf{Z}_t = \mathbf{z}_t)]^{-1}$)

Motivating example - UKCP18 data

Many nuclear facilities built to withstand 10^{-4} **annual exceedance probability events**.

Such events will not be fixed in the non-stationary setting.



Conclusions

- We have developed a **novel modelling framework** for asymptotically independent, non-stationary data structures.
- Can estimate non-stationary return curves that **reflect observed trends**.
- Our work makes a **contribution** to a particularly **sparse field**.

Discussion

- **Uncertainty** - difficult to quantify in any meaningful way.
- **Modelling choices** - quantile levels, number of quantile pairs, degree of polynomial, covariate function forms.
- Assumes we can **perfectly account** for marginal non-stationarity - **never** the case in practice.
- **No theoretical results**.
- Lots of avenues for future research.

Thanks for listening! Does anyone have any questions?

References I

Davison, A. C. and Smith, R. L. (1990). Models for Exceedances Over High Thresholds. *Journal of the Royal Statistical Society: Series B (Methodological)*, 52(3):393–425.

Eastoe, E. F. and Tawn, J. A. (2009). Modelling non-stationary extremes with application to surface level ozone. *Journal of the Royal Statistical Society. Series C: Applied Statistics*, 58(1):25–45.

Knochenhauer, M. and Louko, P. (2004). Guidance for External Events Analysis. In *Probabilistic Safety Assessment and Management*, pages 1498–1503. Springer, London.

Ledford, A. W. and Tawn, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, 83(1):169–187.

Marcon, G., Naveau, P., and Padoan, S. (2017). A semi-parametric stochastic generator for bivariate extreme events. *Stat*, 6(1):184–201.

References II

Murphy-Barltrop, C. J. R. and Wadsworth, J. L. (2022). Modelling non-stationarity in asymptotically independent extremes. *Preprint*.

Wadsworth, J. L. and Tawn, J. A. (2013). A new representation for multivariate tail probabilities. *Bernoulli*, 19(5 B):2689–2714.

Averaging over quantiles

- How do we select q_1 and q_2? **Bias-variance trade-off**
- We instead consider a **range of quantile pairs** simultaneously and compute an **average** estimator
- $\{(q_{1,j}, q_{2,j}) \mid 1 \leq j \leq m\}$ be quantiles near one, with $q_{1,j} < q_{2,j} < 1$

$$\bar{\lambda}_{QR}(w \mid \mathbf{z}_t) := \frac{1}{m} \sum_{j=1}^{m} \hat{\lambda}_j(w \mid \mathbf{z}_t).$$

- Found this estimator to **outperform** individual pairs.

Smoothing

- $\bar{\lambda}_{QR}$ is pointwise for each ray $w \in [0, 1]$ - so **horrible and bumpy**!
- Non-smooth ADF estimates that we would not expect to observe in practice.
- Use parametric polynomial functions to get a **smooth estimate**.
- Bernstein-Bézier polynomials of degree $k > 0$

$$f(w) = \sum_{i=0}^{k} \alpha_i \binom{k}{i} w^i (1 - w)^{k-i}$$

with coefficients $\alpha_i \in [0, 1]$ for each i

Smoothing

- In standard form, this polynomial is **fixed**.
- We extend to allow **covariate influence**.
- Also, $0 \leq f(w) \leq 1$: but λ can be **above** 1!
- Propose family

$$f_t(w) = \sum_{i=0}^{k} \beta_i(\mathbf{z}_t) \binom{k}{i} w^i (1-w)^{k-i} \; : \; \boldsymbol{\beta}(\mathbf{z}_t) \in [0, \infty)^{k+1}$$

  $\beta_i : \mathbb{R}^p \to [0, \infty)$ are positive functions of covariates for all i
- $\beta_0 = \beta_k = 1$.

Smoothing

- **Goal**: estimate f_t for all t.
- We propose **parametric forms** for the coefficient functions - e.g. $\beta_i(\mathbf{z}_t) = \exp(a_i + b_i \mathbf{z}_t)$ with $a_i, b_i \in \mathbb{R}$.
- Find $\hat{a}_i, \hat{b}_i$ for all i by **minimising**

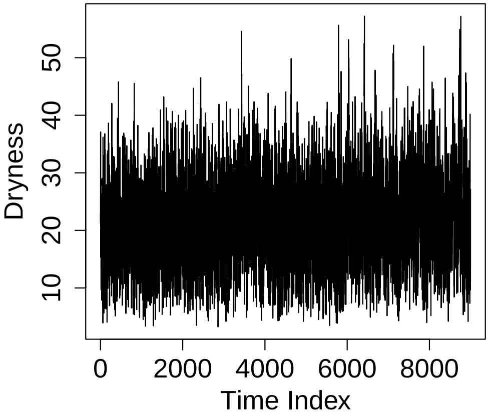
$$\left| \bar{\lambda}_{QR}(w \mid \mathbf{z}_t) - f_t(w) \right|$$

  over all w and t.
- Define resulting estimator to be $\bar{\lambda}_{BP}(\cdot \mid \mathbf{z}_t)$

Theoretical properties.

- We impose some theoretical properties on both estimators.
- Ensures lower bound and endpoints are satisfied.
- $\lambda(w) \geq \max(w, 1 - w)$
- $\lambda(0) = \lambda(1) = 1$

Dryness



Marginal trends