Improving estimation for asymptotically independent bivariate extremes via global estimators for the angular dependence function

Callum Murphy-Barltrop, Jennifer Wadsworth and Emma Eastoe

EVA 2023

- Have just finished my PhD at Lancaster University, supervised by Jennifer Wadsworth and Emma Eastoe.
- Will be starting a 3-year postdoc at TU Dresden this September.

Background

- Multivariate extreme value model $=$ framework for evaluating extremal dependence of a random vector (X, Y).
- Classical approaches based on the framework of multivariate regular variation (MRV).

Background

- Suppose (X, Y) has standard Pareto margins and define $R := X + Y$, $V := X/(X + Y)$
- (X, Y) is MRV if, for any measurable $B \subset [0, 1]$,

$$\lim_{r \to \infty} \Pr(V \in B, R > sr \mid R > r) = H(B)s^{-1}, s \geq 1.$$

- TLDR; as R gets big, V and R become independent.
- H is the spectral measure and summarises the extremal dependence (Resnick, 1987).

Background

- Fundamental classification of extremal dependence given by the coefficient.

$$\chi = \lim_{u \to 1} \Pr(F_Y(Y) > u \mid F_X(X) > u) \in [0, 1].$$

- $\chi = 0 \Rightarrow$ asymptotic independence.
- $\chi > 0 \Rightarrow$ asymptotic dependence.

Background

- Classical approaches based on MRV (most approaches) cannot distinguish between asymptotic independence and complete independence.
- In both cases
$$H(\{0\}) = H(\{1\}) = 0.5.$$
- Consequently, MRV cannot accurately extrapolate for asymptotically independent vectors (Ledford and Tawn, 1996, 1997; Heffernan and Tawn, 2004).

Background

- More recent techniques have been developed that can model both dependence regimes.
- Throughout the remainder of this talk, let (X, Y) be a random vector on standard exponential margins.

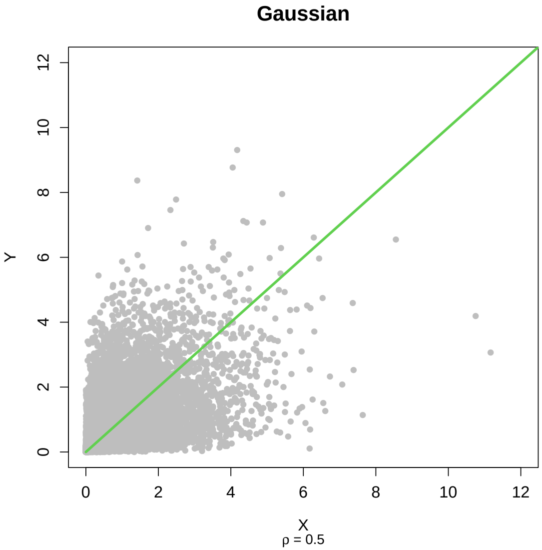
Background

- First approach given by Ledford and Tawn (1996, 1997).

  $$\Pr(X > u, Y > u) = \Pr(\min(X, Y) > u) \to L(e^u)\exp(-u/\eta),$$

  as $u \to \infty$, with L slowly varying and $\eta \in (0, 1]$.
- Asymptotic dependence: $\eta = 1$.
- TLDR; joint survivor function decays exponentially along the $y = x$ line.

Background



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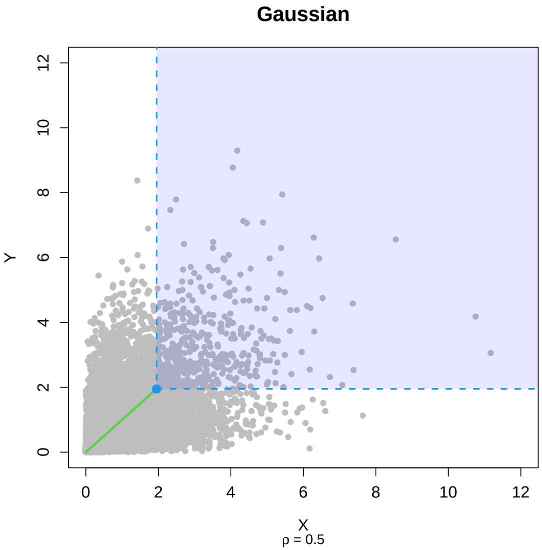


Background

- Ledford and Tawn (1996, 1997) model was extended in Wadsworth and Tawn (2013).
- Given any ray $w \in [0,1]$,

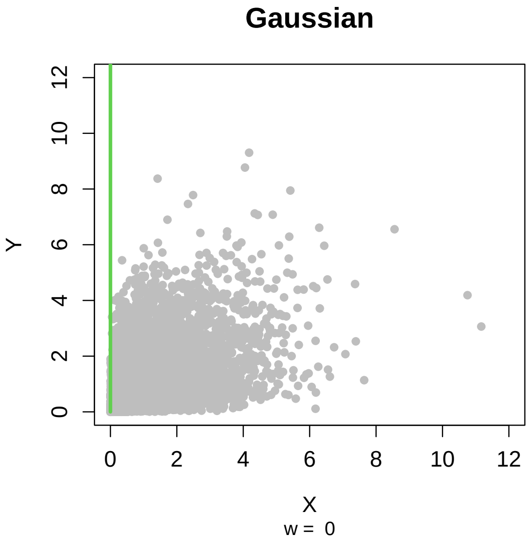
$$\Pr(X > wu, Y > (1-w)u) = \Pr\left(\min\left\{\frac{X}{w}, \frac{Y}{1-w}\right\} > u\right) \to L(e^u \mid w)\exp(-\lambda(w)u),$$

  as $u \to \infty$, with L slowly varying.
- $\lambda(w), w \in [0,1]$ is known as the **angular dependence function** (ADF).

Background

- TLDR; joint survivor function decays exponentially in all regions.
- $\lambda(w) \geq \max(w, 1 - w)$.
- Asymptotic dependence: $\lambda(w) = \max(w, 1 - w)$.
- $\eta = 1/(2\lambda(0.5))$.

Background



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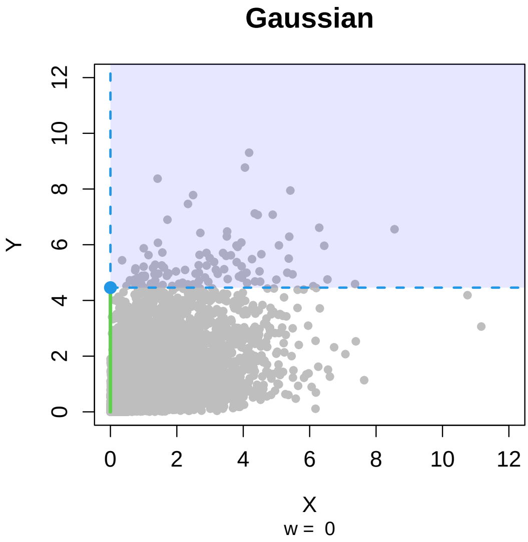
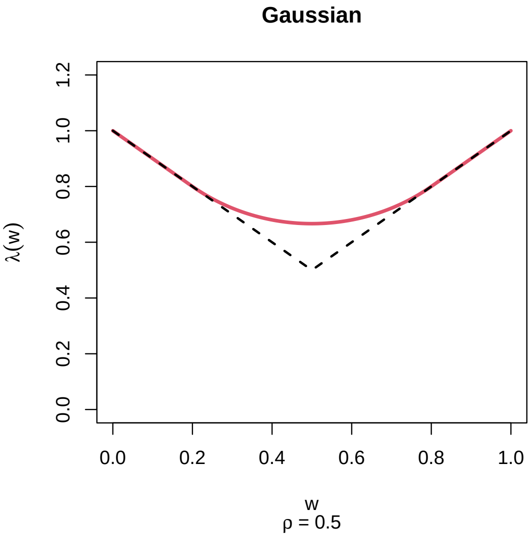


Background



Background

- Alternative representation for multivariate extremes given by Heffernan and Tawn (2004).
- Roughly speaking, they assume that

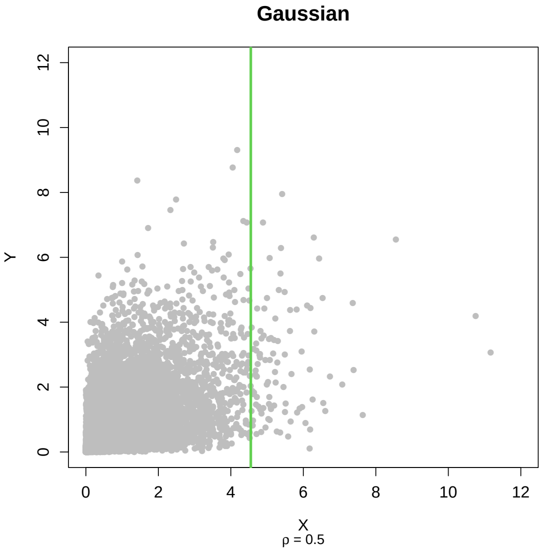
$$(Y \mid X > u) = \alpha_{y|x} X + X^{\beta_{y|x}} Z$$

as $u \to \infty$, where Z is a residual process.

Background

- $\alpha_{y|x} \in [0, 1]$, $\beta_{y|x} \in [0, 1)$.
- Could alternatively condition on $Y > u$.
- Asymptotic dependence: $\alpha_{y|x} = 1$, $\beta_{y|x} = 0$.
- TLDR; fancy regression.

Background



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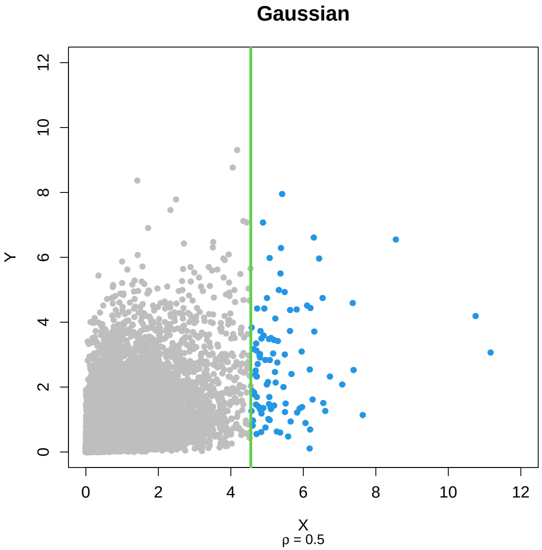

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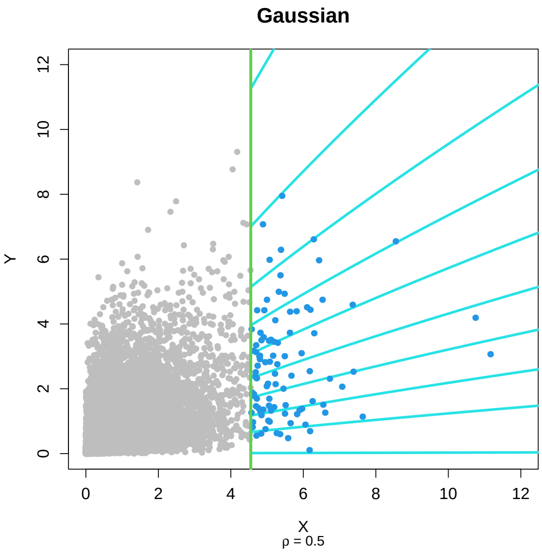


$\alpha_{y|x}$ determines slope.

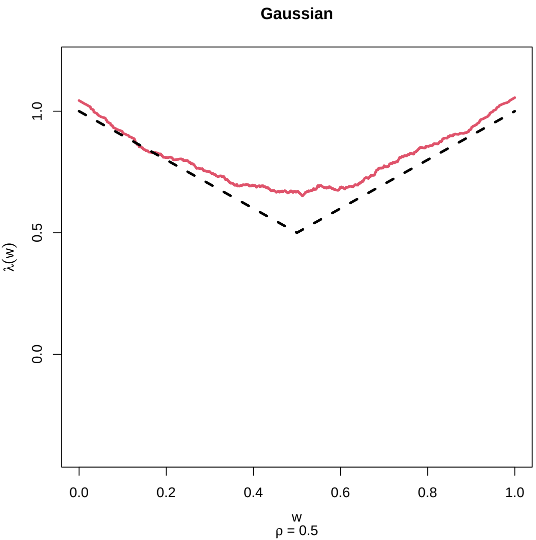
Background

- Heffernan and Tawn (2004) model widely used in practice.
- Few applications of Wadsworth and Tawn (2013) exist, even though this model outperform Heffernan and Tawn (2004) in certain scenarios (Murphy-Barltrop et al., 2023b).

Background

- **The problem**: until recently, the ADF has been estimated in a pointwise manner using the Hill estimator.
- Consider each $w \in [0, 1]$ in turn, obtain an estimate $\hat{\lambda}(w)$, stitch together.
- Results in non-smooth and unrealistic ADF estimates.

Background

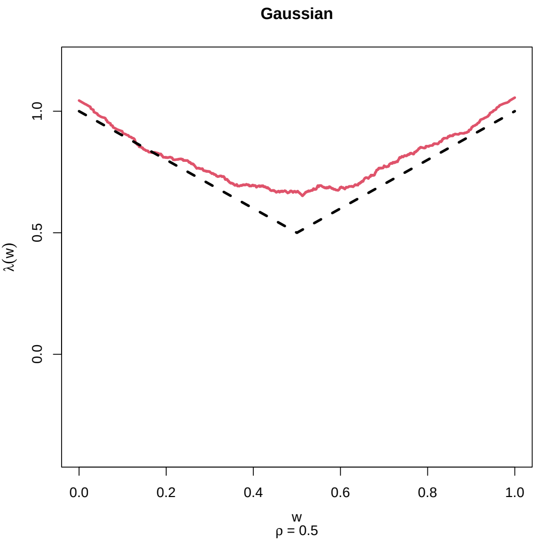


Background

Other issues with this estimator:

- Does not necessarily satisfy endpoint conditions $\lambda(0) = \lambda(1) = 1$.
- Does not necessarily satisfy lower bound $\lambda(w) \geq \max(w, 1 - w), w \in [0, 1]$.

Background



Background

Our goal:

Provide smooth ADF estimators that satisfy theoretical constraints.

Existing ADF estimators

- Denote non-smooth Hill estimator $\hat{\lambda}_H$
- Simpson and Tawn (2022) recently provided the first smooth estimator (others now available).
- This estimator exploits the results of Nolde and Wadsworth (2022).

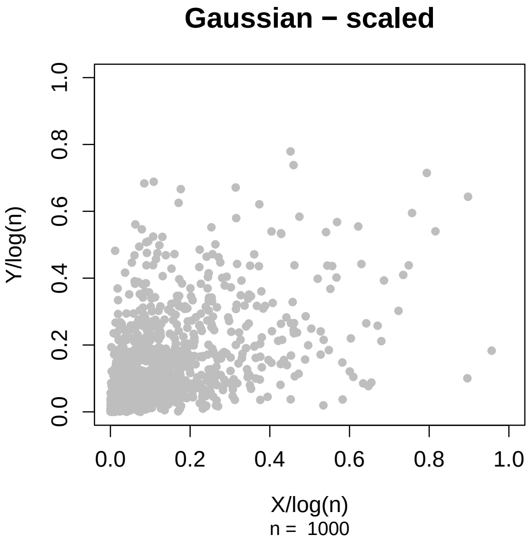
Existing ADF estimators

- Given n independent realisations from (X, Y), we consider the shape of

$$C_n^* := \{(X_i, Y_i)/\log n;\ i = 1, \ldots, n\},$$

as $n \to \infty$.

Existing ADF estimators



Existing ADF estimators

- As $n \to \infty$, we have that C_n^* converges onto the set

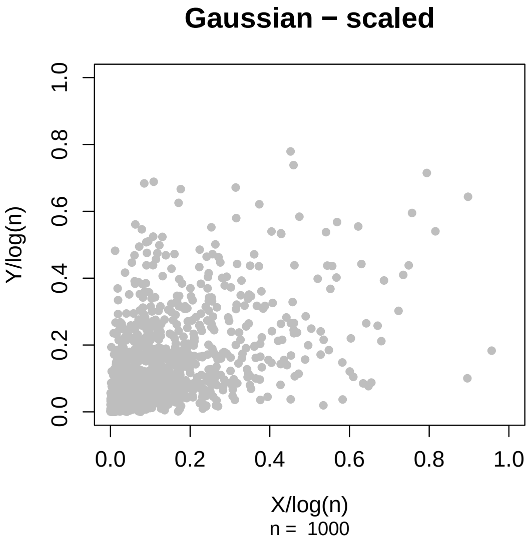
$$G^* = \{(x, y) : g(x, y) \leq 1\} \subseteq [0, 1]^2.$$

  where g is the so-called Gauge function.
- Interest lies in studying the boundary set given by

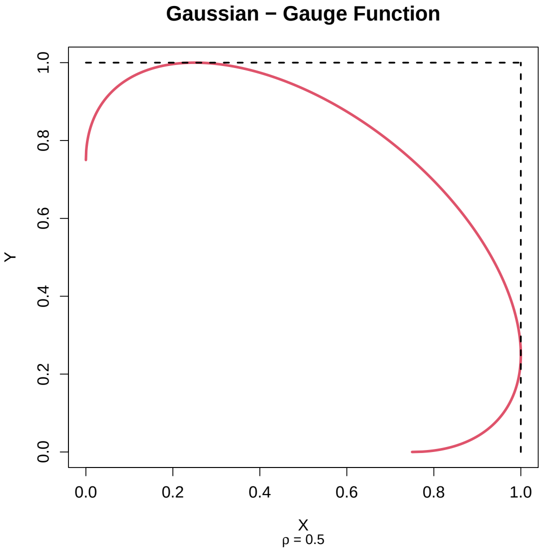
$$G = \{(x, y) : g(x, y) = 1\} \subset [0, 1]^2.$$

- TLDR; scaled data points converge onto sets with nice shapes.

Existing ADF estimators



Existing ADF estimators



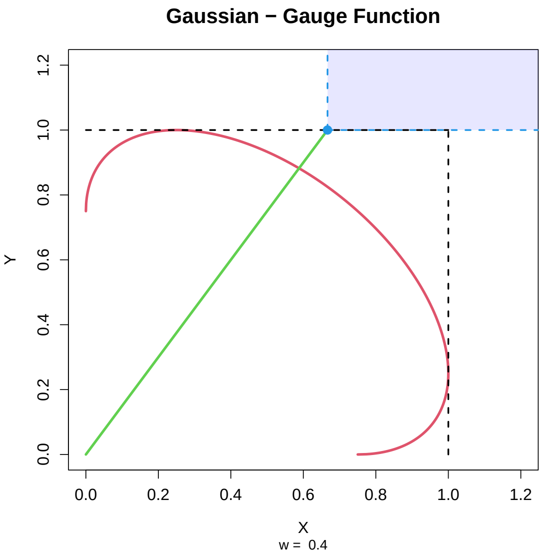
Existing ADF estimators

- The boundary G links the models of Heffernan and Tawn (2004) and Wadsworth and Tawn (2013).

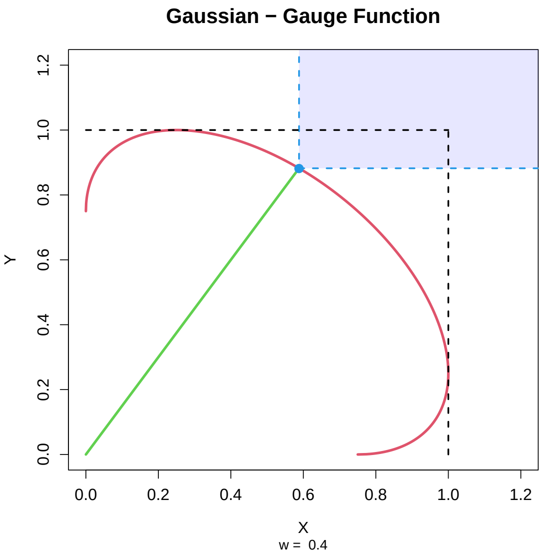
Existing ADF estimators

- We have that
$$\lambda(w) = \frac{\max(w, 1-w)}{s_w},$$
where
$$s_w = \min\left\{s \in [0,1] : sS_w \cap G = \emptyset\right\}.$$

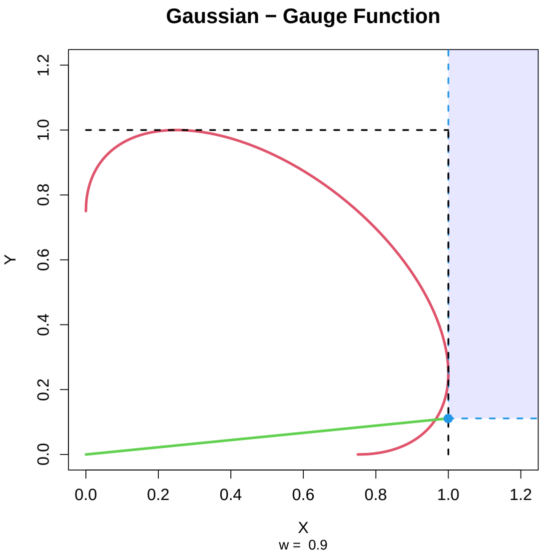
Existing ADF estimators



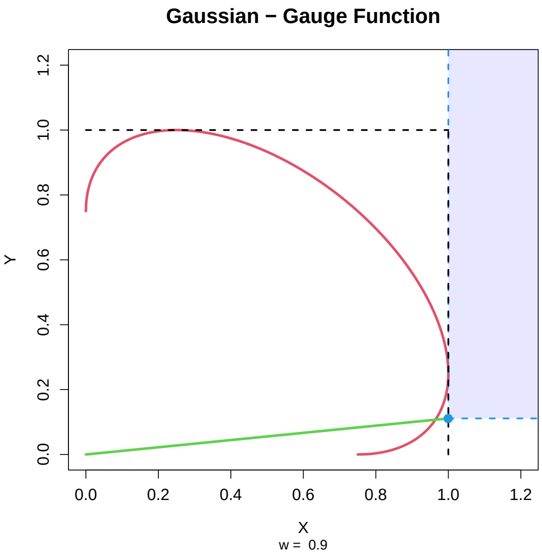
Existing ADF estimators



Existing ADF estimators



Existing ADF estimators



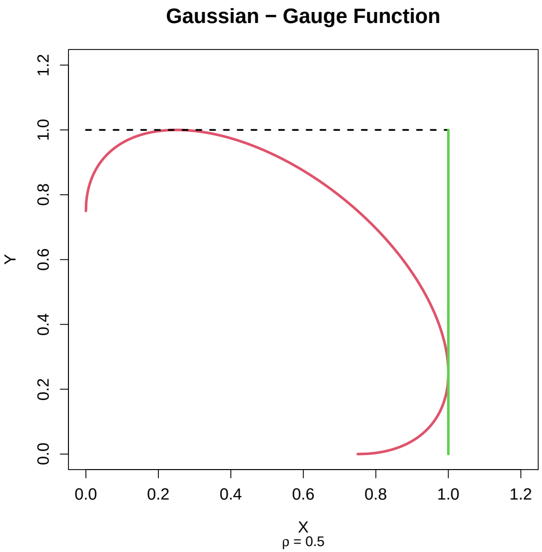
Existing ADF estimators

- Now consider the normalising function $a_{y|x}(x) = \alpha_{y|x}x$ (or $a_{x|y}(x) = \alpha_{x|y}x$).
- We have that

$$\alpha_{y|x} = \max\left\{\tilde{\alpha} \in [0,1] : g\left(1, \tilde{\alpha}\right) = 1\right\}.$$

$$\alpha_{x|y} = \max\left\{\tilde{\alpha} \in [0,1] : g\left(\tilde{\alpha}, 1\right) = 1\right\}.$$

Existing ADF estimators



Existing ADF estimators

Once we have G, we get the rest for free.

- Simpson and Tawn (2022) provided the first approach for estimating G.
- Let $\hat{\lambda}_{ST}$ denote the smooth ADF estimator obtained using this approach.

Novel estimators

- Recall our objective: to provide smooth estimates of the ADF.
- Lots of approaches are available for smooth estimation of the PDF (e.g. Guillotte and Perron, 2016; Marcon et al., 2016).
- Many use flexible smooth Bernstein-Bézier polynomials.

Novel estimators

- Consider this family

$$\mathcal{B}_k^* = \Bigg\{ (1-w)^k + \sum_{i=1}^{k-1} \beta_i \binom{k}{i} w^i (1-w)^{k-i} + w^k =: f(w) \;\Bigg|\; w \in [0,1], \boldsymbol{\beta} \in [0,\infty)^{k-1} \text{ such that } f(w) \geq \max(w, 1-w) \Bigg\}. \quad (1)$$

- Satisfies all theoretical conditions for the ADF.

Novel estimators

- We assume $\lambda \in \mathcal{B}_k^*$, so $\lambda(w) = \lambda(w; \boldsymbol{\beta})$.
- What remains is to estimate $\boldsymbol{\beta} \in [0, \infty)^{k-1}$.
- We achieve this through a composite likelihood approach.

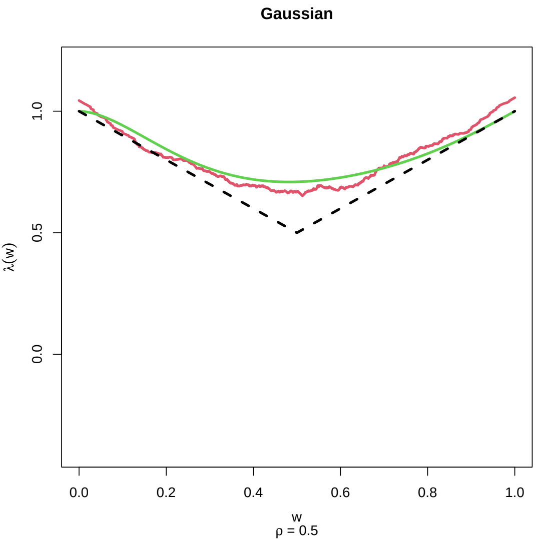
Novel estimators

- We multiply this function over rays (components) $w \in [0, 1]$ to give one overall likelihood function.

$$\mathcal{L}_C(\boldsymbol{\beta}) = \prod_{w \in \mathcal{W}} \prod_{t_w^* \in \mathbf{t}_w^*} \lambda(w;\ \boldsymbol{\beta}) e^{-\lambda(w;\ \boldsymbol{\beta}) t_w^*}.$$

- Let $\hat{\boldsymbol{\beta}}_{CL}$ denote the maximum likelihood estimator of $\boldsymbol{\beta}$.
- Corresponding ADF estimator given by $\hat{\lambda}_{CL}(\cdot) = \hat{\lambda}(\cdot; \boldsymbol{\beta} = \hat{\boldsymbol{\beta}}_{CL})$.

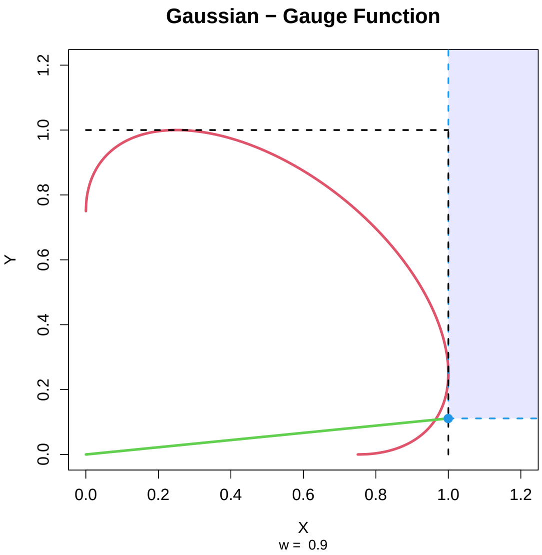
Novel estimators



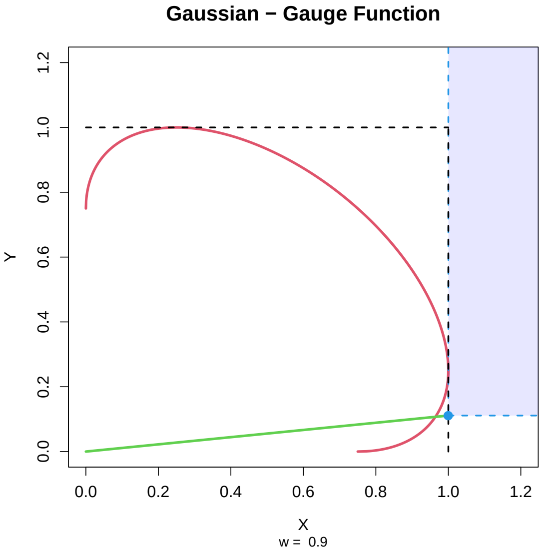
Novel estimators

- We also exploited a corollary from Nolde and Wadsworth (2022) to improve ADF estimation.
- In particular, we found that for all $w \in [0, \alpha_{x|y}/(1+\alpha_{x|y})] \bigcup [1/(1+\alpha_{y|x}), 1]$, $\lambda(w) = \max(w, 1-w)$.
- If we know conditional extremes parameters $\alpha_{x|y}$, $\alpha_{y|x}$, we know where $\lambda(w, 1-w) = \max(w, 1-w)$ (and vice versa).

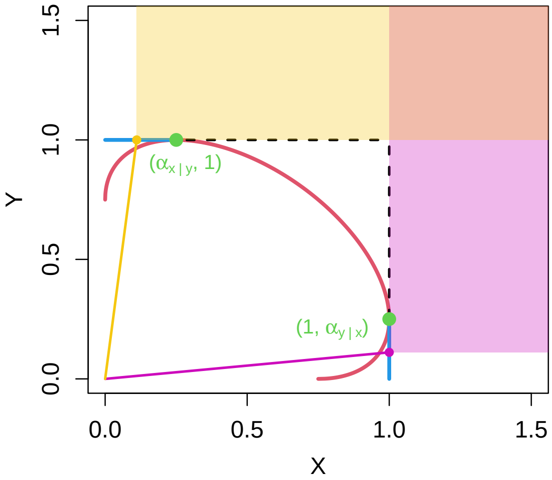
Novel estimators



Novel estimators



Novel estimators



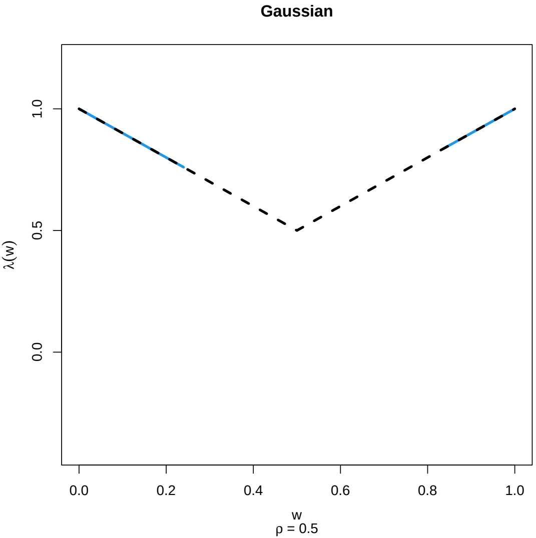
Novel estimators

Procedure:

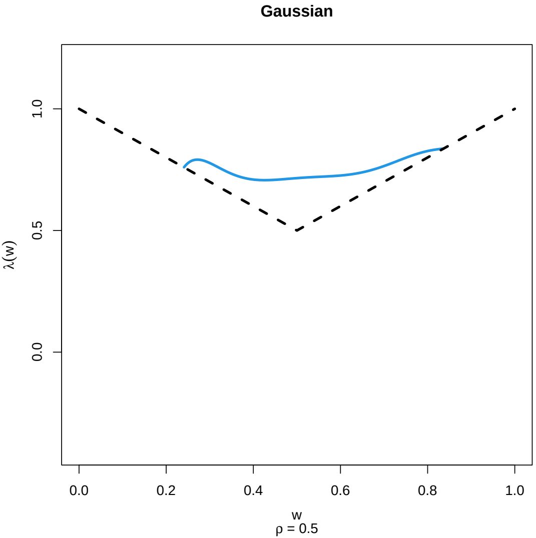
1. Use standard techniques to obtain $\hat{\alpha}_{y|x}$ and $\hat{\alpha}_{x|y}$.
2. Set $\lambda(w) = \max(w, 1 - w)$ for all $w \in [0, \hat{\alpha}_{x|y}/(1 + \hat{\alpha}_{x|y})] \bigcup [1/(1 + \hat{\alpha}_{y|x}), 1]$.
3. Estimate λ using composite likelihood for $w \in (\hat{\alpha}_{x|y}/(1 + \hat{\alpha}_{x|y}), 1/(1 + \hat{\alpha}_{y|x}))$ (after rescaling the polynomial family).

This gives us a second composite likelihood estimator $\hat{\lambda}_{CL2}$.

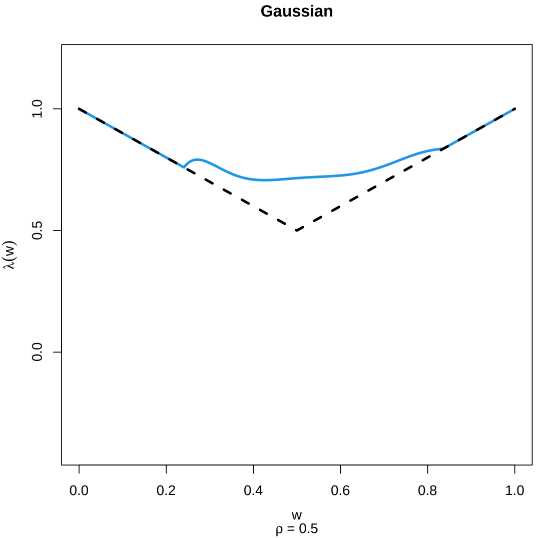
Novel estimators



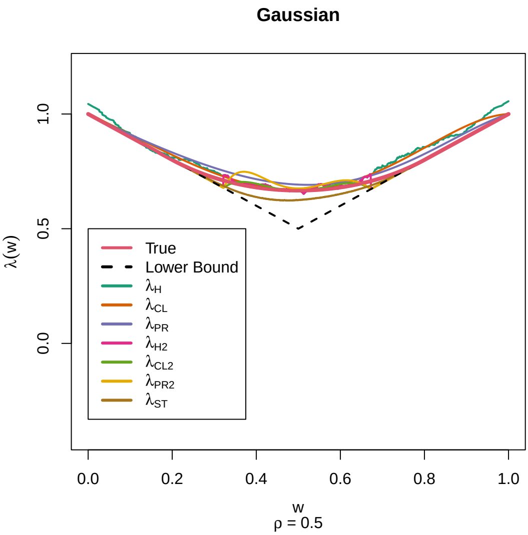
Novel estimators



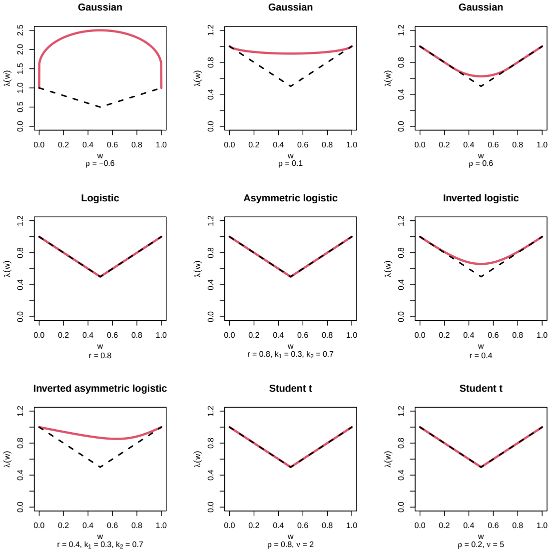
Novel estimators



Novel estimators



Simulation study



Simulation study

Table: RMISE values (multiplied by 100) for each estimator and copula combination. Smallest RMISE values in each row are highlighted in bold, with values reported to 3 significant figures.

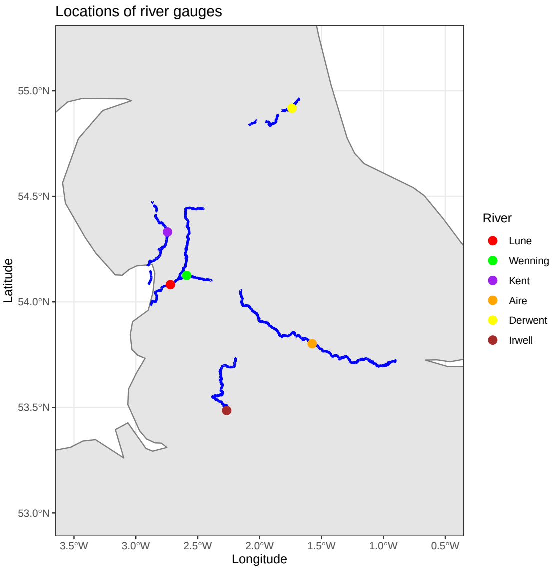
Copula	$\hat{\lambda}_H$	$\hat{\lambda}_{CL}$	$\hat{\lambda}_{PR}$	$\hat{\lambda}_{H2}$	$\hat{\lambda}_{CL2}$	$\hat{\lambda}_{PR2}$	$\hat{\lambda}_{ST}$
Copula 1	**61.1**	61.3	66.2	61.4	61.9	66.7	63.7
Copula 2	3.55	3.33	3.64	3.51	3.33	3.63	**2.95**
Copula 3	3.78	3.48	3.84	3.27	3.22	3.57	**1.09**
Copula 4	4.9	4.79	6.92	4.28	4.25	6.17	**2.77**
Copula 5	14.1	14.1	17.1	14.1	14.1	17	**12.1**
Copula 6	2.51	1.97	2.15	2	**1.74**	1.9	2.12
Copula 7	2.93	**2.64**	2.88	2.87	2.66	2.89	3.96
Copula 8	2.49	2.72	2.95	0.66	**0.6**	0.789	1.87
Copula 9	12.1	12	14.9	12	12	14.9	**11.1**

Simulation study

Global estimators $>$ pointwise estimators

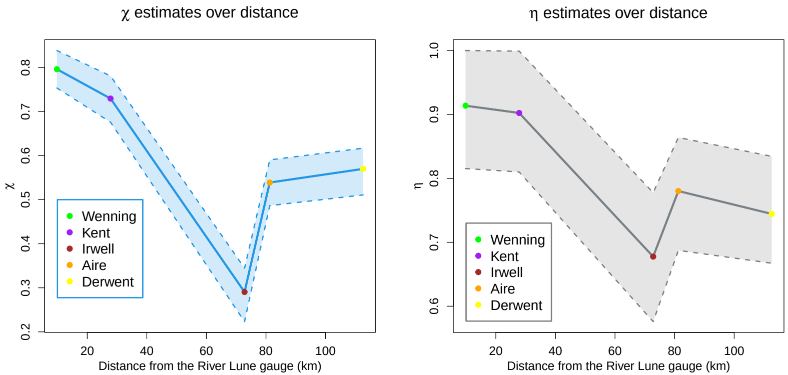
Important insight: estimators linked to Gauge functions/limit sets performed best.

Case study



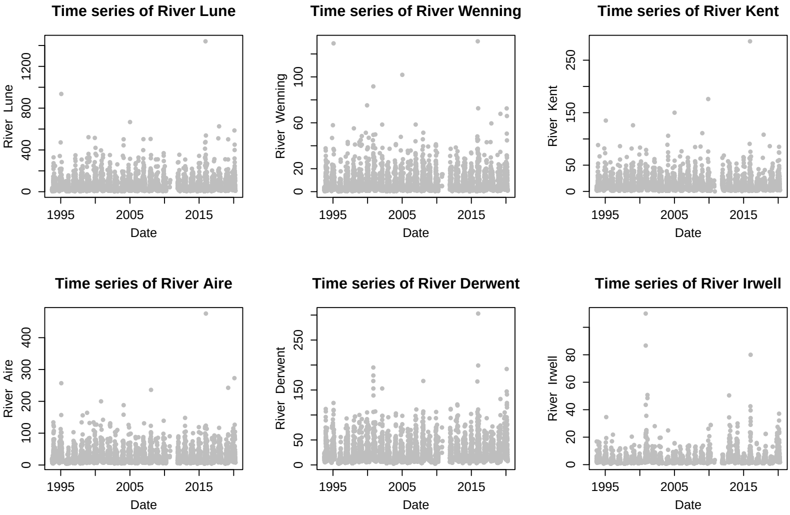
Locations of river gauges
55.0°N
54.5°N
54.0°N
53.5°N
53.0°N
Latitude
3.5°W
3.0°W
2.5°W
2.0°W
1.5°W
1.0°W
0.5°W
Longitude
River
Lune
Wenning
Kent
Aire
Derwent
Irwell

Case study



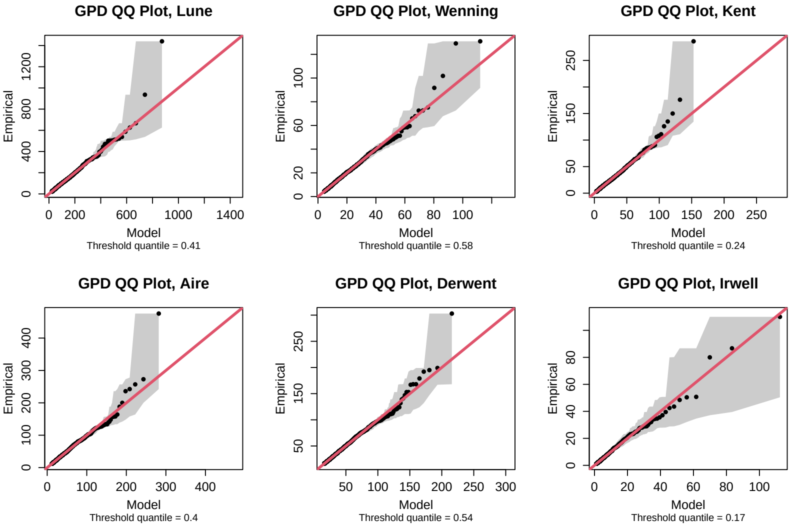
χ estimates over distance
χ
0.2
0.3
0.4
0.5
0.6
0.7
0.8
20
40
60
80
100
Distance from the River Lune gauge (km)
Wenning
Kent
Irwell
Aire
Derwent
η estimates over distance
η
0.6
0.7
0.8
0.9
1.0
20
40
60
80
100
Distance from the River Lune gauge (km)
Wenning
Kent
Irwell
Aire
Derwent

Case study

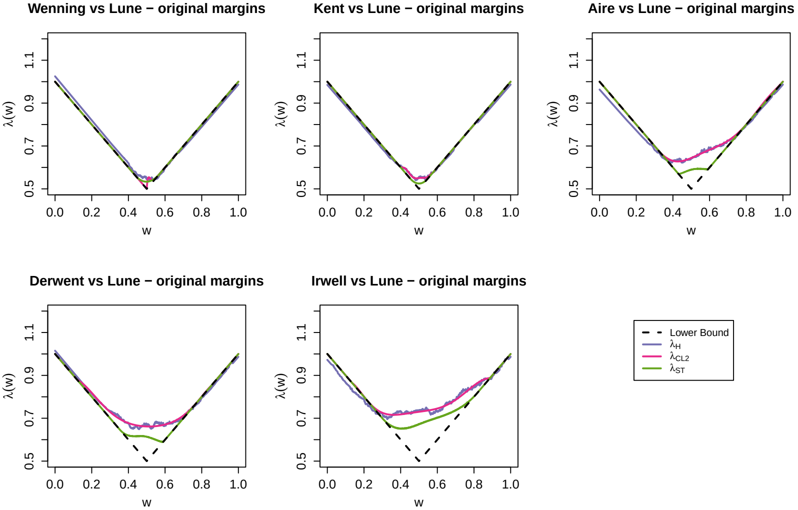


Time series of River Lune
River Lune
0
400
800
1200
1995
2005
2015
Date
Time series of River Wenning
River Wenning
0
20
60
100
1995
2005
2015
Date
Time series of River Kent
River Kent
0
50
150
250
1995
2005
2015
Date
Time series of River Aire
River Aire
0
100
200
300
400
1995
2005
2015
Date
Time series of River Derwent
River Derwent
0
50
150
250
1995
2005
2015
Date
Time series of River Irwell
River Irwell
0
20
40
60
80
1995
2005
2015
Date

Case study


GPD QQ Plot, Lune
Empirical
Model
Threshold quantile = 0.41
GPD QQ Plot, Wenning
Empirical
Model
Threshold quantile = 0.58
GPD QQ Plot, Kent
Empirical
Model
Threshold quantile = 0.24
GPD QQ Plot, Aire
Empirical
Model
Threshold quantile = 0.4
GPD QQ Plot, Derwent
Empirical
Model
Threshold quantile = 0.54
GPD QQ Plot, Irwell
Empirical
Model
Threshold quantile = 0.17

Case study



Wenning vs Lune − original margins
Kent vs Lune − original margins
Aire vs Lune − original margins
Derwent vs Lune − original margins
Irwell vs Lune − original margins
λ(w)
w
0.5
0.7
0.9
1.1
0.0
0.2
0.4
0.6
0.8
1.0
Lower Bound
λ_H
λ_{CL2}
λ_{ST}

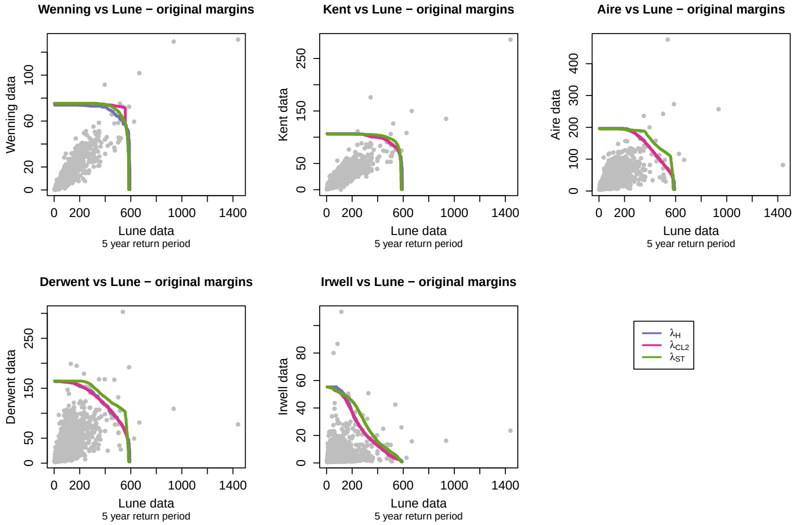
Case Study

Return curves defined by the set

$$\text{RC}(p) := \{(x, y) \in \mathbb{R}^2 \mid \Pr(X > x, Y > y) = p\},$$

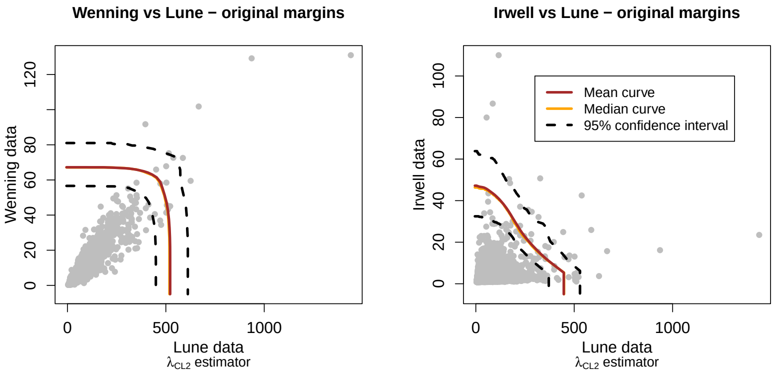
where p is very small.

Case study



Return curve diagnostics indicated good accuracy.

Case study



Wenning vs Lune − original margins
Wenning data
Lune data
λCL2 estimator
0
20
40
60
80
120
0
500
1000
Irwell vs Lune − original margins
Irwell data
Lune data
λCL2 estimator
Mean curve
Median curve
95% confidence interval
0
20
40
60
80
100
0
500
1000

Discussion

In summary

- We have proposed a range of global estimators for the ADF and compared these to existing techniques.
- Global estimators consistently outperform pointwise techniques.
- ADF is a valuable tool for estimation of joint extremes under asymptotic independence.

Discussion

- More generally, this paper illustrates the benefits of applying the limit set representation for multivariate extremes in practice (both directly and indirectly).

Discussion

- Lack of theoretical results for estimators.
- Limited to bivariate setting.
- Selecting tuning parameters (not discussed here).

See Murphy-Barltrop et al. (2023a) for further details.

Thanks for listening!

Does anyone have any questions?

Plus, an EVT joke courtesy of ChatGPT:
Why did the statistician only trust extreme value theory?
Because he knew the average could be mean.

References I

Guillotte, S. and Perron, F. (2016). Polynomial pickands functions. *Bernoulli*, 22:213–241.

Heffernan, J. E. and Tawn, J. A. (2004). A conditional approach for multivariate extreme values. *Journal of the Royal Statistical Society. Series B: Statistical Methodology*, 66:497–546.

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